Sections 4.3 and 4.4

Increasing and Decreasing Functions

We have previously mentioned in class if a function is increasing/decreasing then it's derivative is positive/negative.

Definition 4.3.1. Let f be defined on an interval I. We say f is *increasing* on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say f is *decreasing* on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

We want to be able to determine when a function is increasing/decreasing using the derivative.

Theorem 4.3.2. Suppose f is continuous on $I = [a, b]$ and differentiable on (a, b). If $f'(x) > 0$ for all x in (a, b), then f is increasing on I. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on I.

Proof.

Example 4.3.3. Find the intervals on which the following functions are increasing and on which they are decreasing.

1. $f(x) = 4 - x^2$

2. $f(x) = x^2 \ln x^2 + 1$

We can use increasing/decreasing behavior to identify local extrema, particularly useful if there is difficulty in evaluating, *i.e.* our procedure from Section 4.1 is obstructed.

Theorem 4.3.4 (First Derivative Test)**.** Suppose f is continuous on a closed interval containing a critical point c, and assume f is differentiable on an interval containing c except perhaps at $x = c$.

1. If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.
2. If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.
3. If f' is positive on both sides of c or is negative on both sides, then f has no local extremum at c.

Example 4.3.5. Find the local extrema of $g(x) = x^{2/3}(2 - x)$.

Concavity

Another graphical feature described by derivatives is concavity.

Definition 4.3.6. Let f be differentiable on an open interval I. If f' is increasing on I, then f is *concave up*. If f' is decreasing on I, then f is *concave down.*

If f is continuous at c and f changes concavity at c, then f has an *inflection point* at c.

We can apply Theorem 4.3.2 to the above definition:

Theorem 4.3.7 (Test for Concavity)**.**

1. If $f'' > 0$ on I, then f is concave up on I.
2. If $f'' < 0$ on I, then f is concave down on I.
3. If f'' changes sign at c, then c is an inflection point.

Example 4.3.8. Identify the intervals of concavity and inflection points of $f(x) = \arcsin x$ on $I = (-1, 1)$.

Finally, we can use the second derivative to test for local extrema:

Theorem 4.3.9 (Second Derivative Test)**.** Suppose f'' is continuous on an open interval containing c where $f'(c) = 0$.

1. If $f''(c) > 0$, then f has a local minimum at c.
2. If $f''(c) < 0$, then f has a local maximum at c.
3. If $f''(c) = 0$, then the test is inconclusive.

Example 4.3.10. Find the local extrema of $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$.

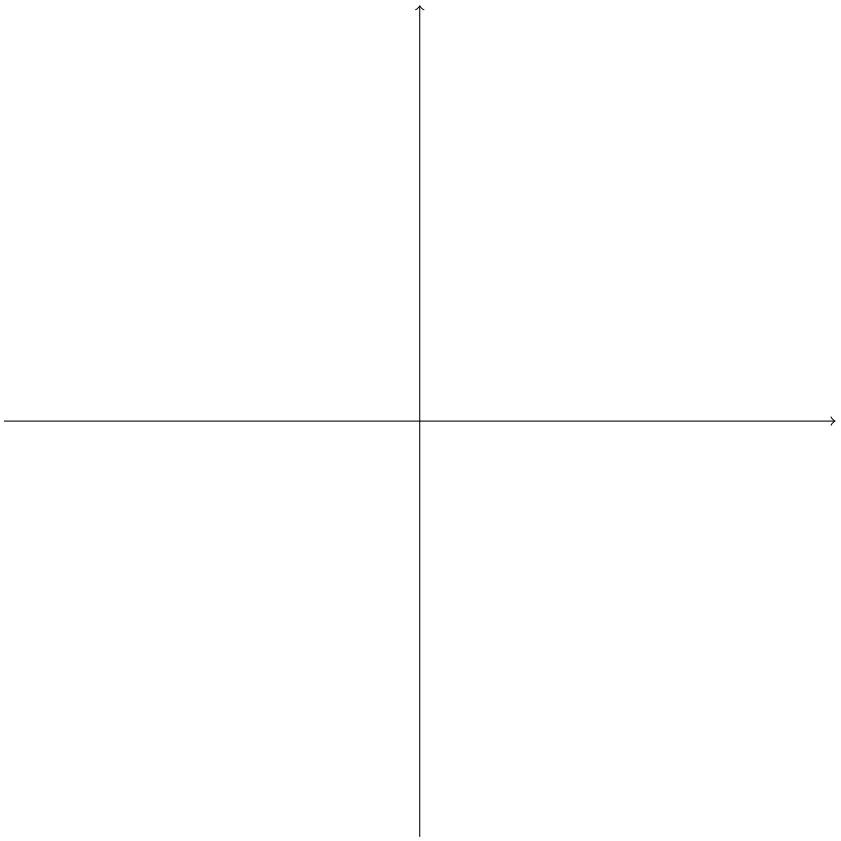
Graphing Functions

In terms of graphing functions, the important features to capture are the roots, the asymptotes, the end-behavior, the critical points, the inflection points, the intervals of increasing/decreasing, and the intervals of concavity. We now have explored all of these topics.

Procedure 4.4.11 (Graphing a Function)**.**

1. Identify the domain.
2. Exploit symmetry: $f(-x) = -f(x)$ or $f(-x) = f(x)$;
   periodicity: $f(x + k) = f(x)$.
3. Compute the first and second derivatives.
4. Find critical and inflection points.
5. Find intervals of increasing/decreasing and concavity.
6. Identify extreme values.
7. Locate asymptotes and determine end behavior: $\lim_{x \to \pm\infty} f(x)$.
8. Find the x- and y-intercepts.
9. Plot graph.

Example 4.4.12. Sketch the graph of $f(x) = \dfrac{10x^3}{x^2 - 1}$.



Example 4.4.13. Sketch the graph of the Gaussian curve: $f(x) = e^{-x^2}$.

